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ON A PROBLEM OF TURÁN ABOUT POLYNOMIALS WITH CURVED MAJORANTS

BY
Q. I. RAHMAN

Abstract. Let $\phi(x)\geqq 0$ for $-1\leqq x\leqq 1$. For a fixed x_0 in $[-1, 1]$ what can be said for max $|p_n'(x_0)|$ if $p_n(x)$ belongs to the class P_ϕ of all polynomials of degree n satisfying the inequality $|p_n(x)|\leqq\phi(x)$ for $-1\leqq x\leqq 1$? The case $\phi(x)=1$ was considered by A. A. Markov and S. N. Bernstein. We investigate the problem when $\phi(x)=(1-x^2)^{1/2}$. We also study the case $\phi(x)=|x|$ and the subclass consisting of polynomials typically real in $|z|<1$.

The following theorem was proved by A. A. Markov in 1889.

Theorem A. *If $p_n(x)$ is a polynomial of degree n, such that $|p_n(x)|\leqq 1$ for $-1\leqq x\leqq 1$, then*

$$\max_{-1\leqq x\leqq 1} |p_n'(x)| \leqq n^2. \tag{1}$$

The original paper [9] of Markov is not readily accessible but an excellent account of this and other related results is presented in [4]. In Theorem A equality is attainable only at ± 1 and only for $p_n(x)=e^{i\gamma}T_n(x)$ where

$$T_n(x) = \cos(n\cos^{-1}x) = 2^{n-1}\prod_1^n\{x-\cos((\nu-\tfrac{1}{2})\pi/n)\}$$
$$= \frac{n}{2}\sum_{m=0}^{[n/2]}(-1)^m\frac{(n-m-1)!}{m!(n-2m)!}(2x)^{n-2m}$$

is the so-called Chebyshev polynomial of the first kind.

For points x lying in the interval $|x|<\cos(\frac{1}{2}\pi/n)$ the following theorem of Bernstein [2] gives a better estimate for $|p_n'(x)|$.

Theorem B. *Under the conditions of Theorem* A

$$|p_n'(x)| \leqq n(1-x^2)^{-1/2}, \qquad -1<x<1. \tag{2}$$

This dominant $n(1-x^2)^{-1/2}$ is the best possible dominant only at the points $x=\cos\{(2k+1)\pi/(2n)\}$, $k=1, 2,\ldots, n-1$. It is, however, asymptotically equal to the precise bound at every fixed point in the interior of the interval as n becomes infinite.

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At a conference on Constructive Function Theory held in Varna, Bulgaria, Professor P. Turán proposed the following problem:

Problem. For any x_0 in $[-1, 1]$ determine max $|p_n'(x_0)|$ for all polynomials $p_n(x)$ of degree $\leqq n$ satisfying the restriction

$$\max_{-1\leqq x\leqq 1} \frac{|p_n(x)|}{\sqrt{(1-x^2)}} = 1. \tag{3}$$

He remarked that even the value of $\max_{-1\leqq x\leqq 1} |p_n'(x)|$ did not seem to be known for the class in (3).

For real-valued polynomials the hypothesis says that the graph of $p_n(x)$ on the interval $-1<x<1$ is contained in the closed unit disk.

Let π_n denote the class of polynomials $p_n(x)$ of degree n which satisfy $|p_n(x)| \leqq (1-x^2)^{1/2}$ for $-1<x<1$.

In looking for the maximum of $|p_n^{(k)}(z^*)|$ (the kth derivative at a given point) for all polynomials $p_n(x)$ of degree at most n which satisfy $|p_n(x)|\leqq 1$ for $-1\leqq x\leqq 1$ it is enough to consider the subclass A_n whose members are in addition real on the real axis. Let $p_n^{(k)}(z^*)=e^{i\gamma}|p_n^{(k)}(z^*)|$ and let $e^{-i\gamma}p_n(z)=p_{n,1}(z)+ip_{n,2}(z)$ where $p_{n,1}$ and $p_{n,2}$ are elements of A_n. Since $p_{n,1}^{(k)}(z^*)=|p_n^{(k)}(z^*)|$ the maximum of $|p_n^{(k)}(z^*)|$ is attained, if at all, for some p_n in A_n. The following theorem of Duffin and Schaeffer [7, p. 240] shows that the functions in A_n are uniformly bounded on every compact set. Hence [1, p. 216] the maximum of $|p_n^{(k)}(z^*)|$ is, in fact, attained.

THEOREM C. *Let $p_n(z)$ be a polynomial of degree n or less such that in the real interval $(-1, 1)$ $|p_n(z)|\leqq 1$. Then for z lying on the ellipse $\mathscr{E}_R$ with foci at -1, $+1$ and semiaxes $\frac{1}{2}(R+R^{-1})$, $\frac{1}{2}(R-R^{-1})$, we have $|p_n(z)|\leqq\frac{1}{2}(R^n+R^{-n})$.*

For precisely the same reason as above the maximum of $|p_n^{(k)}(z^*)|$ over the class π_n is attained for a polynomial which is real on the real axis.

We prove

THEOREM 1. *If $p_n(x)$ is a polynomial of degree n such that $|p_n(x)|\leqq(1-x^2)^{1/2}$ for $-1<x<1$, then*

$$\max_{-1\leqq x\leqq 1} |p_n'(x)| \leqq 2(n-1). \tag{4}$$

If

$$U_n(x) = (1-x^2)^{-1/2}\sin\{(n+1)\cos^{-1}x\} = \sum_{m=0}^{[n/2]}(-1)^m\frac{(n-m)!}{m!(n-2m)!}(2x)^{n-2m}$$

is the nth Chebyshev polynomial of the second kind then $p_n(x)=(1-x^2)U_{n-2}(x)$ satisfies the conditions of Theorem 1 and $|p_n'(\pm 1)|=2(n-1)$. Hence the result is best possible.

THEOREM 2. *Under the conditions of Theorem 1*

$$|p_n'(x)| \leqq \{x^2(1-x^2)^{-1}+(n-1)^2\}^{1/2}, \qquad -1<x<1. \tag{5}$$

The example $p_n(x)=e^{i\gamma}(1-x^2)U_{n-2}(x)$ shows that in (5) equality can be attained at those points of the interval $-1<x<1$ where $(n-1)(1-x^2)^{1/2}\tan\{(n-1)\cos^{-1}x\}=x$.

It follows from Theorem B that if $p_n(x)$ is a polynomial of degree n such that $|p_n(x)|\leqq 1$ for $-1\leqq x\leqq 1$ then $n^{-1}(1-x^2)p_n'(x)\in\pi_{n+1}$. Hence we have the following corollary of Theorem 2.

COROLLARY 1. *If $p_n(x)$ is a polynomial of degree n such that $|p_n(x)|\leqq 1$ for $-1\leqq x\leqq 1$, then*

$$(6)\qquad |(1-x^2)p_n''(x)-2xp_n'(x)| \leqq n\{x^2(1-x^2)^{-1}+n^2\}^{1/2}, \qquad -1<x<1.$$

When $x=0$ inequality (5) *may be restated as follows:*

If $p_n(x)=\sum_{k=0}^{n} a_kx^k\in\pi_n$ then $|a_1|=|p_n'(0)|\leqq n-1$. This inequality is sharp for odd n. Here again the extremal polynomial is $e^{i\gamma}(1-x^2)U_{n-2}(x)$.

We also estimate $|a_2|$.

THEOREM 3. *If $p_n(x)=\sum_{k=0}^{n} a_kx^k\in\pi_n$ then*

$$(7)\qquad |a_2| \leqq \{(n-1)^2+1\}/2.$$

For even n the bound in (7) is attained when $p_n(x)=e^{i\gamma}(1-x^2)U_{n-2}(x)$.

The next theorem is a refinement of Theorem C for polynomials belonging to π_n.

THEOREM 4. *If $p_n(z)\in\pi_n$ then for z lying on the ellipse $\mathscr{E}_R$ with foci at the points -1, $+1$ and semiaxes $\frac{1}{2}(R+R^{-1})$, $\frac{1}{2}(R-R^{-1})$, we have*

$$(8)\qquad |p_n(z)| \leqq |1-z^2|^{1/2}\tfrac{1}{2}(R^{n-1}+R^{-n+1}).$$

The problem of Turán mentioned earlier is a special case of the following general question subsequently asked by him:

Let $\phi(x)\geqq 0$ for $-1\leqq x\leqq 1$. For a fixed x_0 in $[-1, 1]$ what can be said for $\max|p_n'(x_0)|$ if $p_n(x)$ belongs to the class P_ϕ of all polynomials of degree $\leqq n$ satisfying the inequality $|p_n(x)|\leqq\phi(x)$ for $-1\leqq x\leqq 1$.

We shall consider only the simple class $P_{|x|}$ of polynomials $p_n(x)$ of degree $\leqq n$ which are dominated by the function $|x|$ on $[-1, 1]$.

If $p_n(z)$ is a polynomial of degree n such that $|p_n(x)|\leqq|x|$ for $-1\leqq x\leqq 1$ then $p_n(z)=zg_{n-1}(z)$ where $g_{n-1}(z)$ is a polynomial of degree $n-1$. Since, clearly, $|g_{n-1}(x)|\leqq 1$ for $-1\leqq x\leqq 1$, Theorem A gives

$$|p_n'(x)| \leqq |x|\,|g_{n-1}'(x)|+|g_{n-1}(x)| \leqq (n-1)^2+1$$

for $-1\leqq x\leqq 1$. Thus we have

THEOREM 5. *If $p_n(z)$ is a polynomial of degree n such that $|p_n(x)|\leqq|x|$ for $-1\leqq x\leqq 1$ then*

$$(9)\qquad \max_{-1\leqq x\leqq 1}|p_n'(x)| \leqq (n-1)^2+1.$$

The example $p_n(z)=zT_{n-1}(z)$, where $T_{n-1}(z)$ is the Chebyshev polynomial of the first kind of degree $n-1$, shows that (9) is sharp.

We also prove

THEOREM 6. *If $p_n(x)$ is a polynomial of degree $\leqq n$ such that $|p_n(x)|\leqq|x|$ for $-1\leqq x\leqq 1$ then for a fixed x_0 in $(-1, 1)$ we have*

$$|p_n'(x_0)| \leqq \{(n-1)^2x_0^2(1-x_0^2)^{-1}+1\}^{1/2}. \tag{10}$$

The example $p_n(z)=e^{i\gamma}zT_{n-1}(z)$ shows that in (10) equality is attained at those points of the interval $-1<x<1$ where $(1-x^2)^{1/2}\tan\{(n-1)\cos^{-1}x\}=(n-1)x$.

From a geometric point of view, those members of the class $P_{|x|}$ which are typically real [10] in $|z|<1$ constitute an interesting subclass. If we restrict ourselves to this subclass we can replace (9) by a considerably stronger inequality.

THEOREM 7. *Let $p_n(z)$ be a polynomial of degree n such that $|p_n(x)|\leqq|x|$ for $-1\leqq x\leqq 1$. If $p_n(z)$ is typically real in $|z|<1$ then*

$$|p_n'(x)| \leqq (n+1)/2 \tag{11}$$

for $-1\leqq x\leqq 1$.

A function $g(z)$ analytic in $|z|<1$ is typically real in $|z|<1$ if and only if [10, p. 112] $g(z)$ is real for real z and $\operatorname{Re}\{((1-z^2)/z)g(z)\}\neq 0$ in $|z|<1$. Hence if n is odd the polynomial

$$p_n(z) = 2(z+z^3+\cdots+z^n)/(n+1)$$

is typically real in $|z|<1$. It also belongs to $P_{|x|}$, and since $|p_n'(1)|=(n+1)/2$ the bound in (11) cannot in general be improved.

Lemmas. We shall now collect some results which we shall use in the proofs of the above theorems.

If $p_n(x)$ is a polynomial of degree n then $p_n(\cos\theta)$ is a trigonometric polynomial of degree n. Since $d\theta=-(1-x^2)^{-1/2}\,dx$, inequality (2) states that $|(d/d\theta)p_n(\cos\theta)|\leqq n$. Hence Theorem B is a consequence of the following result known as Bernstein's theorem for trigonometric polynomials.

THEOREM D. *If $t(\theta)$ is a trigonometric polynomial of degree n and $|t(\theta)|\leqq 1$, then $|t'(\theta)|\leqq n$.*

It has been remarked by Boas [4, p. 169] that Markov's theorem (Theorem A) would also follow from Bernstein's theorem for trigonometric polynomials if it could be shown that $|p_n'(x)|$ attains its maximum at ± 1 if $p_n(x)$ is extremal.

We observe that the following result which is a refined version of Bernstein's theorem for trigonometric polynomials and which was independently proved by Szegö [12, p. 69], Boas [3, p. 287], van der Corput and Schaake [6, p. 321] is more appropriate for the study of polynomials on the unit interval and gives Markov's theorem as an immediate corollary.

LEMMA 1. *Let* $t(\theta)=\sum_{k=-n}^{n} a_k e^{ik\theta}$ *be a real trigonometric polynomial of degree n. If* $|t(\theta)|\leqq 1$ *then*

$$n^2\{t(\theta)\}^2+\{t'(\theta)\}^2 \leqq n^2. \tag{12}$$

This result plays a central role in our paper. First of all we use it to prove:

LEMMA 2. *If* $p_{n-1}(x)$ *is a real valued polynomial of degree* $n-1$ *such that* $(1-x^2)^{1/2}|p_{n-1}(x)|\leqq 1$ *for* $-1<x<1$ *then*

$$|p_{n-1}(x)| \leqq n \quad \textit{for } -1 \leqq x \leqq 1. \tag{13}$$

Our hypothesis implies that $(\sin\theta)p_{n-1}(\cos\theta)$ is a real trigonometric polynomial of degree n whose absolute value does not exceed 1. Hence according to Lemma 1

$$n^2\sin^2\theta\{p_{n-1}(\cos\theta)\}^2+\{(\cos\theta)p_{n-1}(\cos\theta)+(\sin\theta)(d/d\theta)p_{n-1}(\cos\theta)\}^2 \leqq n^2$$

for real θ. At a point where $|p_{n-1}(\cos\theta)|$ attains its maximum value, $(d/d\theta)p_{n-1}(\cos\theta)$ must vanish. Consequently, at such a point θ_0,

$$n^2(\sin^2\theta_0)\{p_{n-1}(\cos\theta_0)\}^2+(\cos^2\theta_0)\{p_{n-1}(\cos\theta_0)\}^2 \leqq n^2$$

or

$$(n^2-1)(\sin^2\theta_0)\{p_{n-1}(\cos\theta_0)\}^2+\{p_{n-1}(\cos\theta_0)\}^2 \leqq n^2.$$

Therefore $|p_{n-1}(\cos\theta_0)|\leqq n$ which gives the desired result.

For the sake of completeness we include a proof of Lemma 1. In this way we will also be giving a complete and independent proof of Markov's theorem (since it follows from Theorem B in conjunction with Lemma 2). It may be noted that our proof of Lemma 1 depends only on the maximum modulus principle and the Gauss-Lucas theorem [8, p. 84].

Proof of Lemma 1. Let $p(z)$ be a polynomial of degree m such that $|p(z)|\leqq M$ for $|z|\leqq 1$. Then for $|\lambda|>1$ the polynomial $P(z)=p(z)-\lambda M$ does not vanish in $|z|\leqq 1$. Let

$$Q(z) = z^m\overline{P(1/\bar{z})} = z^m\overline{p(1/\bar{z})}-\bar{\lambda}Mz^m = q(z)-\bar{\lambda}Mz^m.$$

Since the function $Q(z)/P(z)$ is holomorphic in $|z|\leqq 1$ and $|Q(z)|=|P(z)|$ for $|z|=1$ it follows from the maximum modulus principle that $|Q(z)/P(z)|\leqq 1$ for $|z|\leqq 1$. Replacing z by $1/\bar{z}$ we conclude that $|P(z)|\leqq|Q(z)|$ for $|z|\geqq 1$. Thus for $|\mu|>1$ all the zeros of the polynomial $P(z)-\mu Q(z)$ lie in $|z|<1$ and so do the zeros of $P'(z)-\mu Q'(z)$ by Gauss-Lucas theorem. Consequently,

$$|p'(z)| = |P'(z)| \leqq |Q'(z)| = |q'(z)-\bar{\lambda}Mmz^{m-1}| \quad \text{for } |z| \geqq 1. \tag{14}$$

According to our hypothesis $|p(z)|\leqq M$ for $|z|\leqq 1$. Since

$$p(z) \equiv z^m\overline{q(1/\bar{z})},$$

we have

$$|z^m\overline{q(1/\bar{z})}| \leqq M \quad \text{for } |z| \leqq 1,$$

i.e., $|q(z)| \leqq M|z|^m$ for $|z| \geqq 1$. Hence for $|\Lambda| > 1$ all the zeros of the polynomial $q(z) - \Lambda M z^m$ lie in $|z| < 1$ and by the Gauss-Lucas theorem so do the zeros of $q'(z) - \Lambda M m z^{m-1}$. This implies that $|q'(z)| \leqq Mm|z|^{m-1}$ for $|z| \geqq 1$. Given a point z in the circular domain $|z| \geqq 1$, this inequality permits us to choose arg λ in (14) such that $|q'(z) - \bar{\lambda} M m z^{m-1}| = |\lambda| Mm|z|^{m-1} - |q'(z)|$. We readily obtain

$$(15) \qquad |p'(z)| + |q'(z)| \leqq Mm|z|^{m-1} \quad \text{for } |z| \geqq 1.$$

In particular, we have

$$(16) \qquad |(d/d\theta)p(e^{i\theta})| + |-imp(e^{i\theta}) + (d/d\theta)p(e^{i\theta})| \leqq Mm.$$

If $t(\theta) = \sum_{k=-n}^{n} a_k e^{ik\theta}$ is a trigonometric polynomial of degree n and $|t(\theta)| \leqq 1$ then $e^{in\theta} t(\theta) = p(e^{i\theta})$ where $p(z)$ is a polynomial of degree $2n$ such that $|p(z)| \leqq 1$ for $|z| \leqq 1$. From (16) we get $|int(\theta) + t'(\theta)| + |-int(\theta) + t'(\theta)| \leqq 2n$. Hence $n^2\{t(\theta)\}^2 + \{t'(\theta)\}^2 \leqq n^2$ if the trigonometric polynomial is real.

For the proof of Theorem 7 we shall need the following result due to de Bruijn [5, Theorem 4].

LEMMA 3. *Let C be a circular domain in the z-plane, and S an arbitrary point set in the w-plane. If the polynomial $P(z)$ of degree n satisfies $P(z) = w \in S$ for any $z \in C$, then we have, for any $z \in C$ and any $\xi \in C$,*

$$(\xi/n)P'(z) + P(z) - zP'(z)/n \in S.$$

Proofs of the theorems.

Proof of Theorem 1. As remarked earlier there is no loss of generality in assuming $p_n(x)$ to be real-valued. Since $p_n(x)$ vanishes at the points -1, $+1$ we have $p_n(x) = (1-x^2)q_{n-2}(x)$ where $q_{n-2}(x)$ is a polynomial of degree $n-2$. We set $(1-x^2)^{1/2}q_{n-2}(x) = f(x)$ and write $p_n(x)$ as the product of $(1-x^2)^{1/2}$ and $f(x)$. Thus

$$(17) \qquad \begin{aligned} |p_n'(x)| &= |-x(1-x^2)^{-1/2}f(x) + (1-x^2)^{1/2}f'(x)| \\ &\leqq |x|\,|(1-x^2)^{-1/2}f(x)| + |(1-x^2)^{1/2}f'(x)|. \end{aligned}$$

We observe that $f(\cos\theta)$ is a trigonometric polynomial of degree $n-1$ whose absolute value does not exceed 1. Hence according to Theorem D $|(d/d\theta)f(\cos\theta)| \leqq n-1$ for real θ. Since

$$(-\sin\theta)\frac{d}{d(\cos\theta)}f(\cos\theta) = \frac{d}{d\theta}f(\cos\theta)$$

we get

$$(18) \qquad |(1-x^2)^{1/2}f'(x)| \leqq n-1 \quad \text{for } -1 \leqq x \leqq 1.$$

Now let us note that $(1-x^2)^{-1/2}f(x)$ is a polynomial of degree $n-2$ such that $|(1-x^2)^{1/2}\cdot(1-x^2)^{-1/2}f(x)| = |f(x)| \leqq 1$ for $-1 \leqq x \leqq 1$. Hence by Lemma 2

$$(19) \qquad |(1-x^2)^{-1/2}f(x)| \leqq n-1 \quad \text{for } -1 \leqq x \leqq 1.$$

Using (18) and (19) in (17) we get the desired estimate for $\max_{-1\leq x\leq 1} |p_n'(x)|$.

REMARK. Our proof of Theorem 1 makes particular use of the fact that the polynomial $p_n(z)$ under consideration vanishes at the points -1, $+1$. However, the bound for $\max_{-1\leq x\leq 1} |p_n'(x)|$ is not very much improved if we only add this requirement to the hypothesis in Markov's theorem. By considering the polynomial $p_n(x)=\cos n \cos^{-1} (x \cos (\pi/2n))$ we see that $\max_{-1\leq x\leq 1} |p_n'(x)|$ can be as large as $n \cot (\pi/2n)$ if $p_n(\pm 1)=0$ and $\max_{-1\leq x\leq 1} |p_n(x)|=1$. A theorem of Schur [11, pp. 284–285] says that $\max_{-1\leq x\leq 1} |p_n'(x)|\leqq n \cot (\pi/2n)$ for every polynomial of degree $\leqq n$ satisfying the inequality $|p_n(x)|\leqq 1$ for $-1\leqq x\leqq 1$ and vanishing at the points -1, $+1$.

Proof of Theorem 2. Without loss of generality we may assume $p_n(x)$ to be real-valued. Again setting $f(x)=(1-x^2)^{-1/2}p_n(x)$ we see that $f(\cos\theta)$ is a real trigonometric polynomial of degree $n-1$ whose absolute value does not exceed 1. Hence from Lemma 1

$$(n-1)^2f^2(x)+(1-x^2)\{f'(x)\}^2 \leqq (n-1)^2 \quad \text{for } -1 \leqq x \leqq 1. \tag{20}$$

Using this inequality in (17) we conclude that for $-1<x<1$

$$\begin{aligned} |p_n'(x)| &\leqq |x|(1-x^2)^{-1/2}|f(x)|+(n-1)\{1-|f(x)|^2\}^{1/2} \\ &\leqq \max_{0\leq y\leq 1} \{|x|(1-x^2)^{-1/2}y+(n-1)(1-y^2)^{1/2}\}. \end{aligned}$$

For a given x in $(-1, 1)$ the maximum of the expression $|x|(1-x^2)^{-1/2}y+(n-1)(1-y^2)^{1/2}$ is $\{x^2(1-x^2)^{-1}+(n-1)^2\}^{1/2}$ which is attained when

$$y = |x|\{(n-1)^2(1-x^2)+x^2\}^{-1/2}.$$

Proof of Theorem 3. We have

$$|a_2| = \tfrac{1}{2}|p_n''(0)| = \tfrac{1}{2}|f''(0)-f(0)| \leqq \tfrac{1}{2}\{|f''(0)|+|f(0)|\}$$

where $f(x)=(1-x^2)^{-1/2}p_n(x)$. Now if $F(\theta)=f(\cos\theta)$ then $|f''(0)|=|F''(\pi/2)|$ and hence by Theorem D $|f''(0)|\leqq(n-1)^2$. Since $|f(0)|\leqq 1$ we get the desired result.

Proof of Theorem 4. This result is proved in exactly the same way as Theorem C was proved by Duffin and Schaeffer [7, p. 240]. We need only to observe that $f(\cos z)=(\operatorname{cosec} z)p_n(\cos z)$ is an entire function of exponential type $n-1$.

If $p_n(z)=(1-z^2)U_{n-2}(z)$, then (8) becomes an equality at the points

$$z = ((R+R^{-1})/2) \cos \phi_k \pm i((R-R^{-1})/2) \sin \phi_k$$

where $\phi_k=\{(2k+(-1)^k)/2(n-1)\}\pi$, $k=0, 1, 2, \ldots$.

Proof of Theorem 6. It is enough to prove the theorem for polynomials which assume real values on the real axis. We have $p_n(x)=xg_{n-1}(x)$ where $g_{n-1}(x)$ is a polynomial of degree $n-1$ which assumes real values for real x and $|g_{n-1}(x)|\leqq 1$ for $-1\leqq x\leqq 1$. Thus $g_{n-1}(\cos\theta)$ is a real trigonometric polynomial of degree $n-1$ such that $|g_{n-1}(\cos\theta)|\leqq 1$. Hence from Lemma 1 we get

$$(n-1)^2\{g_{n-1}(x)\}^2+(1-x^2)\{g_{n-1}'(x)\}^2 \leqq (n-1)^2 \quad \text{for } -1 \leqq x \leqq 1. \tag{21}$$

We use this inequality in $|p_n'(x)| \leqq |g_{n-1}(x)| + |x|\,|g_{n-1}'(x)|$ to complete the proof of the theorem in precisely the same way as for Theorem 2.

Proof of Theorem 7. According to hypothesis $p_n(z)$ assumes real values in $|z|<1$ if and only if z is real. Hence $p_n'(x) \neq 0$ for $-1<x<1$ and $p_n(x)$ is a monotonic function on the interval $-1 \leqq x \leqq 1$. Without loss of generality we may suppose $p_n(x)$ to be increasing on $[-1, 1]$. Let x_0 be a given point of the open interval $(0, 1)$. The polynomial $P(z)=p_n(x_0 z)$ is typically real in $|z|<1/x_0$ and hence in $|z| \leqq 1$. Also $|P(x)| \leqq x_0$ for $-1 \leqq x \leqq 1$. Since the only zero of $P(z)$ in $|z|<|x_0|^{-1}$ is a simple zero at the origin, $Q(z)=z^n P(z^{-1})$ is a polynomial of degree $n-1$ having all its zeros in $|z| \leqq x_0$. Hence according to Walsh's generalization of Laguerre's theorem [13, Lemma 1, p. 13] $Q'(1)/Q(1)=(n-1)/(1-w)$ where $|w| \leqq x_0$. Consequently $|Q'(1)| \geqq ((n-1)/(1+x_0))|Q(1)|$, i.e.

$$|np_n(x_0)-x_0 p_n'(x_0)| \geqq \frac{n-1}{1+x_0}|p_n(x_0)| = \frac{n-1}{1+x_0}p_n(x_0). \tag{22}$$

But if $z_1, z_2, \ldots, z_{n-1}$ are the zeros of $z^{-1}p_n(z)$ then

$$\frac{x_0 p_n'(x_0)}{p_n(x_0)} = \operatorname{Re}\left\{\frac{x_0 p_n'(x_0)}{p_n(x_0)}\right\} = 1+\sum_{j=1}^{n-1} \operatorname{Re}\left(\frac{x_0}{x_0-z_j}\right)$$

where $\operatorname{Re}(x_0/(x_0-z_j)) \leqq \frac{1}{2}$, $1 \leqq j \leqq n-1$, since $|z_j| \geqq 1$. Thus $np_n(x_0)-x_0 p_n'(x_0) \geqq \frac{1}{2}(n-1)p_n(x_0) \geqq 0$, and (22) can be written as

$$np_n(x_0)-x_0 p_n'(x_0) \geqq \frac{n-1}{1+x_0}p_n(x_0).$$

This implies that the point $p_n(x_0)-n^{-1}x_0 p_n'(x_0)$ lies on the interval

$$[(1-n^{-1})(1+x_0)^{-1}p_n(x_0), p_n(x_0)].$$

Now we note that the image S of the circular domain $|z| \leqq x_0$ under the mapping $w=p_n(z)$ lies in the plane cut along the positive real axis from $p_n(x_0)$ to infinity. According to Lemma 3 the disk

$$|w-\{p_n(x_0)-n^{-1}x_0 p_n'(x_0)\}| \leqq n^{-1}x_0 p_n'(x_0)$$

lies in S. This is possible only if

$$n^{-1}x_0 p_n'(x_0) \leqq p_n(x_0)\{1-(1-n^{-1})(1+x_0)^{-1}\}.$$

Since $p_n(x_0) \leqq x_0$ we get

$$p_n'(x_0) \leqq n-(n-1)(1+x_0)^{-1}. \tag{23}$$

By continuity $p_n'(0) \leqq 1$ and $p_n'(1) \leqq (n+1)/2$. Hence

$$\max_{0 \leqq x \leqq 1} p_n'(x) \leqq (n+1)/2. \tag{24}$$

Applying this result to $-p_n(-x)$ we get

$$\max_{-1 \leqq x \leqq 0} p_n'(x) \leqq (n+1)/2. \tag{25}$$

The desired result follows from (24) and (25).

Inequality (23) gives an estimate for $|p_n'(x_0)|$ at a fixed point x_0 in $[-1, 1]$ but the bound does not appear to be sharp except at -1, 0, $+1$.

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Université de Montréal, Montréal, Quebec, Canada